SOLUTIONS OF FRACTIONAL HEAT-AND WAVE-LIKE EQUATIONS VIA FRACTIONAL VARIATIONAL ITERATION METHOD

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Abstract

We propose a new reliable fractional variational iteration method (FVIM) for fractional nonlinear partial differential equations. The iteration procedure is based on Jumarie’s fractional derivative approach. Several examples have been solved to elucidate the effectiveness of the proposed method and the results are compared with the exact solution, revealing high accuracy and efficiency.

1. Introduction

In last two decades, fractional differential equations have gained much interest due to exact description of nonlinear phenomena in fluid
flow, seismology, biology, chemistry, economic, engineering, and other areas of science. However, the fractional calculus is three centuries old as the conventional calculus. The aim of the present work is to construct exact solutions of nonlinear time-fractional equations.

The fractional partial differential equations appear very frequently in physical sciences. In the past century, notable contributions [1-25] have been made to both the theory and applications of the fractional differential equations. These equations are increasingly used to model problems in research areas as diverse as dynamical systems, mechanical systems, control, chaos, chaos synchronization, continuous-time random walk, anomalous diffusive and sub diffusive systems, unification of diffusion and wave propagation phenomenon and others. The most important advantage of using fractional differential equations in these and other applications is their non-local property. It is well known that the integer-order differential operator is a local operator, but the fractional order differential operator is a non-local operator. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is more realistic and it is one reason why fractional calculus has become more and more popular.

In this study, we apply the variational iteration method (VIM) to obtain analytical solution for nonlinear fractional initial value problem. Recently, a promising analytical technique called variational iteration method proposed by He [1] has successfully been applied to solve many types of linear and nonlinear functional equations. The basic inspiration of this paper is the extension of variational iteration method using modified Riemann-Liouville derivative to find analytical solutions to time fractional heat- and wave- like equation with variable coefficients.

2. Basic Definitions

We give some basic definitions, notations, and properties of the fractional calculus theory, which are used further in this paper.
Definition 1. Assume \( f : R \to R, x \to f(x) \) denote a continuous (but not necessarily differentiable) function and let the partition \( h > 0 \) in the interval \([0, 1] \). Jumarie’s derivative is defined through the fractional difference

\[
\Delta^\alpha = (FW - 1)^\alpha f(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x + (\alpha - k)),
\]

where \( FWf(x) = f(x + h) \). Then, the fractional derivative (Jumarie [13]) is defined as the following limit:

\[
f^{(\alpha)} = \lim_{h \to 0} \frac{\Delta^\alpha [f(x) - f(0)]}{h^\alpha}.
\]

This definition is close to the standard definition of derivative, and as a direct result, the \( \alpha \)-th derivative of a constant \( 0 < \alpha < 1 \); is zero.

Definition 2. The Riemann-Liouville fractional integral operator of order \( \alpha \geq 0 \) for a function \( f \in C_\mu, \mu \geq -1 \) is defined as

\[
I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - \xi)^{\alpha-1} f(\xi) d\xi, \quad \alpha > 0, \quad t > 0.
\]

Definition 3. The Jumarie’s modified Riemann-Liouville derivative (Jumarie [13]) is defined as

\[
I_x^\alpha f(x) = \frac{1}{\Gamma(m - \alpha)} \frac{d^m}{dx^m} \int_0^x (x - \xi)^{m-\alpha} (f(\xi) - f(0)) d\xi,
\]

where \( x \in [0, 1], \quad m - 1 < \alpha \leq m, \quad m \geq 1 \).

The proposed modified Riemann-Liouville derivative as shown in Equation (4) is strictly equivalent to Equation (2). Meanwhile, we would introduce some properties of the fractional modified Riemann-Liouville derivative.
(a) Fractional Leibnitz product law:

\[ D_x^\alpha (u v) = u^{(\alpha)} v + u v^{(\alpha)}. \]  

(b) Fractional Leibnitz formulation:

\[ I_x^\alpha D_x^\alpha f(x) = f(x) - f(0), \quad 0 < \alpha \leq 1. \]

Therefore, the integration by part can be used during the fractional calculus

\[ I_x^\alpha (u^{(\alpha)} v) = (uv)^{b}_{a} - I_x^\alpha u v^{(\alpha)}. \]

**Definition 4.** Fractional derivative of compounded functions is defined as

\[ d^\alpha f(x) \equiv \Gamma(1 + \alpha) df, \quad 0 < \alpha < 1. \]

**Definition 5.** The integral with respect to \((d\xi)^\alpha\) is defined as the solution of fractional differential equation given by equation

\[ dy \equiv f(x) (dx)^\alpha, \quad x \geq 0, \quad y(0) = 0, \quad 0 < \alpha < 1, \]

\[ y \equiv \int_0^x f(\xi) (d\xi)^\alpha = \alpha \int_0^x (x - \xi)^{\alpha - 1} f(\xi) d\xi, \quad 0 < \alpha \leq 1. \]

For example, \(f(x) = x^\beta\) in Equation (10), one obtains

\[ \int_0^x \xi^\beta (d\xi)^\alpha = \frac{\Gamma(1 + \alpha)\Gamma(1 + \beta)}{\Gamma(1 + \alpha + \beta)} x^{\alpha + \beta}, \quad 0 < \alpha \leq 1. \]

**Definition 6.** Assume that the continuous function \(f : \mathbb{R} \to \mathbb{R}, x \to f(x)\) has a fractional derivative of order \(k\alpha\), for any positive integer \(k\) and any \(\alpha; 0 < \alpha \leq 1\), then the following equality holds, which is

\[ f(x + h) = \sum_{k=0}^{\infty} \frac{h^{ak}}{ak!} f^{(ak)}(x), \quad 0 < \alpha \leq 1. \]
On making the substitution $h \to x$ and $x \to 0$, we obtain the fractional Mc-Laurin series

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\alpha k!} f^{(\alpha k)}(0), \quad 0 < \alpha \leq 1. \quad (13)$$

### 3. Fractional Variational Iteration Method (FVIM)

In order to elucidate the solution procedure of the VIM, we consider the following fractional differential equation:

$$\frac{\partial^{\alpha} u(x, t)}{\partial t^{\alpha}} = K[x]u(x, t) + q(x, t), \quad t > 0, \quad x \in R, \quad (14)$$

with the initial condition

$$u(x, 0) = f(x),$$

where $K[x]$ is the differential operator in $x$, $f(x)$ and $q(x, t)$ are continuous functions. According to VIM introduced by He [1], we can construct a correction functional for Equation (14) as follows:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \left[ \lambda \left( \frac{\partial^{\alpha} u_n}{\partial t^{\alpha}} - K[x]u(x, t) - q(x, t) \right) \right] d\xi, \quad (15)$$

Combining Equations (10) and (15), we obtained a proposed correction functional

$$u_{n+1}(x, t) = u_n(x, t) + \frac{1}{\Gamma(\alpha + 1)} \int_0^t \lambda(\xi) \left( \frac{\partial^{\alpha} u_n}{\partial \xi^{\alpha}} (x, \xi) - K[x]u(x, \xi) - q(x, \xi) \right) d\xi^{\alpha}. \quad (16)$$
It is obvious that the successive approximation \( u_j, j \geq 0 \) can be established by determining \( \lambda \). A general Lagrange’s multiplier, which can be identified optimally via the variational theory. The function \( \tilde{u}_n \) is a restricted variation, which means \( \delta \tilde{u}_n = 0 \). Therefore, we first determine Lagrange’s multiplier that will be identified optimally via integration by parts. The successive approximation of the \( u_{n+1}(x, t), \ n \geq 0 \) solution \( u(x, t) \) will be readily obtained upon using the Lagrange’s multiplier and by using any selective function \( u_0 \). The initial values are usually used for selecting the zeroth approximation \( u_0 \). With \( \lambda \) determined, several approximations \( u_j, j \geq 0 \) follows immediately. Consequently, the exact solution may be obtained by using the following equation:

\[
u(x, t) = \lim_{n \to 0} u_n(x, t)\]  

(17)

4. Numerical Examples

Example 4.1. We consider the one-dimensional fractional heat-like equation

\[
D_t^\alpha u(x, t) = \frac{1}{2} x^2 u_{xx}, \quad t > 0, \quad 0 < x < 1, \quad 0 < \alpha \leq 1.
\]  

(18)

Subject to the boundary conditions

\[
u(0, t) = 0, \quad u(1, t) = e^t,
\]

and the initial condition

\[
u(x, 0) = x^2.
\]

Then a corrected functional for Equation (18) can be constructed as follows:

\[
u_{n+1}(x, t) = \nu_n(x, t) - \frac{1}{\Gamma(1 + \alpha)} \int_0^t \left( \frac{\partial^\alpha u_n}{\partial \xi^\alpha} - \frac{1}{2} x^2 \frac{\partial^2 u_n}{\partial x^2} \right) (d\xi)^\alpha.
\]  

(19)
We start with an initial approximation

\[ u_0(x, t) = x^2. \]

The other components can be determined by using above iteration formula as

\[ u_1(x, t) = \frac{x^2}{\Gamma(1 + \alpha)} t^\alpha, \]
\[ u_2(x, t) = \frac{x^2}{\Gamma(1 + 2\alpha)} t^{2\alpha}, \]
\[ u_3(x, t) = \frac{x^2}{\Gamma(1 + 3\alpha)} t^{3\alpha}, \]

and so on.

Consequently, we have the following series solution:

\[ u(x, t) = x^2 + \frac{x^2}{\Gamma(1 + \alpha)} t^\alpha + \frac{x^2}{\Gamma(1 + 2\alpha)} t^{2\alpha} + \frac{x^2}{\Gamma(1 + 3\alpha)} t^{3\alpha} + \ldots \] (20)

Immediately upon replacing \( \alpha \) by 1 in Equation (20), which is exactly the same as solution obtained in [22].

\[ u(x, t) = x^2 e^t. \]

**Example 4.2.** We consider the two-dimensional fractional heat-like equation

\[ D_t^\alpha u(x, t) = u_{xx} + u_{yy}, \quad t > 0, \quad 0 < x, y < 1, \quad 0 < \alpha \leq 1, \quad \ldots \] (21)

Subject to the boundary conditions

\[ u(0, y, t) = 0, \quad u(2\pi, y, t) = 0, \]
\[ u(x, 0, t) = 0, \quad u(x, 2\pi, t) = 0, \]

and the initial condition

\[ u(x, y, 0) = \sin x \sin y. \]
Then a corrected functional for Equation (21) can be constructed as follows:

\[ u_{n+1}(x, y, t) = u_n(x, y, t) - \frac{1}{\Gamma(1 + \alpha)} \int_0^t \left\{ \frac{\partial^\alpha u_n}{\partial \xi^\alpha} - u_{nx} + u_{ny} \right\} (d \xi)^\alpha. \]  

(22)

We start with an initial approximation

\[ u_0(x, y, t) = \sin x \sin y. \]

The other successive approximations are obtained as follows:

\[ u_1(x, y, t) = -\frac{\sin x \sin y}{\Gamma(1 + \alpha)} t^\alpha, \]
\[ u_2(x, y, t) = \frac{4 \sin x \sin y}{\Gamma(1 + 2\alpha)} t^{2\alpha}, \]
\[ u_3(x, y, t) = -\frac{8 \sin x \sin y}{\Gamma(1 + 3\alpha)} t^{3\alpha}, \]

and so on.

The series solution is given by

\[ u(x, y, t) = \sin x \sin y - \frac{\sin x \sin y}{\Gamma(1 + \alpha)} t^\alpha + \frac{4 \sin x \sin y}{\Gamma(1 + 2\alpha)} t^{2\alpha} \]
\[ - \frac{8 \sin x \sin y}{\Gamma(1 + 3\alpha)} t^{3\alpha} + \ldots, \]

(23)

replacing \( \alpha \) by 1 in Equation (23), the exact solution is the same as solution obtained in [22].

\[ u(x, y, t) = e^{-2t} \sin x \sin y. \]

**Example 4.3.** Consider the three-dimensional fractional heat-like equation

\[ D_x^\alpha u(x, t) = x^4 y^4 z^4 + \frac{1}{36} \left[ x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz} \right], \]

\[ t > 0, \quad 0 < x, y, z < 1, \quad 0 < \alpha < 1. \]

(24)
Subject to the boundary conditions

\[ u(0, y, z, t) = 0, \quad u(1, y, z, t) = y^4 z^4 \left( e^t - 1 \right), \]
\[ u(x, 0, z, t) = 0, \quad u(x, 1, z, t) = x^4 z^4 \left( e^t - 1 \right), \]
\[ u(x, y, 0, t) = 0, \quad u(x, y, 1, t) = x^4 y^4 \left( e^t - 1 \right), \]

and the initial condition

\[ u(x, y, z, 0) = 0. \]

The iteration formula for Equation (24) is given by

\[
u_{n+1}(x, y, z, t) = u_n(x, y, z, t) - \frac{1}{\Gamma(1 + \alpha)}
\]
\[
\times \int_0^t \left[ \frac{\partial^\alpha u_n}{\partial \xi^\alpha} - x^4 y^4 z^4 - \frac{1}{36} \left[ x^2 u_{nxx} + y^2 u_{nyy} + z^2 u_{nzz} \right] \right] (d\xi)^\alpha.
\]

We start with an initial approximation

\[ u_0(x, y, z, t) = \frac{x^4 y^4 z^4}{\Gamma(1 + \alpha)} t^\alpha. \]

We obtain the following successive approximations as follows:

\[ u_1(x, y, z, t) = \frac{x^4 y^4 z^4}{\Gamma(1 + 2\alpha)} t^{2\alpha}, \]
\[ u_2(x, y, z, t) = \frac{x^4 y^4 z^4}{\Gamma(1 + 3\alpha)} t^{3\alpha}, \]
\[ u_3(x, y, z, t) = \frac{x^4 y^4 z^4}{\Gamma(1 + 4\alpha)} t^{4\alpha}, \]

and so on.
The series solution is given by

\[ u(x, y, z, t) = \sum_{\alpha=0}^{\infty} \frac{x^\alpha y^\alpha z^\alpha}{\Gamma(1+\alpha)} t^\alpha + \frac{x^\alpha y^\alpha z^\alpha}{\Gamma(1+3\alpha)} t^{3\alpha} + \frac{x^\alpha y^\alpha z^\alpha}{\Gamma(1+4\alpha)} t^{4\alpha} + \ldots \] (25)

Immediately upon replacing \( \alpha \) by 1 in Equation (25), which is exactly the same as solution obtained in [22],

\[ u(x, y, z, t) = x^4 y^4 z^4 (e^t - 1). \]

**Example 4.4.** Next, we consider the following two-dimensional fractional wave-like equation:

\[ D_t^\alpha u(x, t) = \frac{1}{12} (x^2 u_{xx} + y^2 u_{yy}), \]

\[ t > 0, \quad 0 < x, y < 1, \quad 1 < \alpha \leq 2. \] (26)

Subject to the boundary conditions

\[ u(0, y, t) = 0, \quad u(1, y, t) = 4 \cosh t, \]

\[ u(x, 0, t) = 0, \quad u(x, 1, t) = 4 \sinh t, \]

and the initial condition

\[ u(x, y, 0) = x^4, \quad u_t(x, y, 0) = y^4. \]

The exact solution (\( \alpha = 2 \)) was found to be [22]

\[ u(x, y, t) = x^4 \cosh t + y^4 \sinh t. \]

To solve the above Equation (26) by means of aforesaid method, we have the following iteration formula:

\[ u_{n+1}(x, y, t) = u_n(x, y, t) - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left( \frac{\partial^\alpha}{\partial \xi^\alpha} u_n - \frac{1}{12} (x^2 u_{nxx} + y^2 u_{nyy}) \right) (d\xi)^\alpha. \] (27)
To get the iteration, we start with an initial approximation as
\[ u_0(x, y, t) = x^4 + y^4 t. \]

We obtain the following successive approximations as follows:
\[ u_1(x, y, t) = \frac{x^4}{\Gamma(1 + \alpha)} t^\alpha + \frac{y^4}{\Gamma(2 + \alpha)} t^{1+\alpha}, \]
\[ u_2(x, y, t) = \frac{x^4}{\Gamma(1 + 2\alpha)} t^{2\alpha} + \frac{y^4}{\Gamma(2 + 2\alpha)} t^{1+2\alpha}. \]

The series solution is given as
\[ u(x, y, t) = x^4 \left[ 1 + \frac{1}{\Gamma(1 + \alpha)} t^\alpha + \frac{1}{\Gamma(1 + 2\alpha)} t^{2\alpha} + \ldots \right] \]
\[ + y^4 \left[ t + \frac{1}{\Gamma(2 + \alpha)} t^{1+\alpha} + \frac{1}{\Gamma(2 + 2\alpha)} t^{1+2\alpha} + \ldots \right], \]
upon replacing \( \alpha \) by 2, which is exactly the same as solution obtained in [22].

**Example 4.5.** Finally, we consider the following three-dimensional fractional wave-like equation:
\[ D_t^\alpha u(x, t) = x^2 + y^2 + z^2 + \frac{1}{2} \left( x^2 u_{xx} + y^2 u_{yy} + z^2 u_{zz} \right), \]
\[ t > 0, \quad 0 < x, y, z < 1, \quad 1 < \alpha < 2. \]  

Subject to the boundary conditions
\[ u(0, y, z, t) = y^2 (e^t - 1) + z^2 (e^t - 1), \quad u(1, y, z, t) = (1 + y^2) (e^t - 1) + z^2 (e^t - 1), \]
\[ u(x, 0, z, t) = x^2 (e^t - 1) + z^2 (e^t - 1), \quad u(x, 1, z, t) = (1 + x^2) (e^t - 1) + z^2 (e^t - 1), \]
\[ u(x, y, 0, t) = x^2 (e^t - 1) + y^2 (e^t - 1), \quad u(x, y, 1, t) = (1 + x^2) (e^t - 1) + y^2 (e^t - 1), \]
and the initial condition
\[ u(x, y, z, 0) = 0, \quad u_t(x, y, z, 0) = x^2 + y^2 - z^2. \]
To solve the above Equation (28) by means of aforesaid method, we have the following iteration formula:

\[ u_{n+1}(x, y, z, t) = u_n(x, y, z, t) - \frac{1}{\Gamma(1 + \alpha)} \int_0^t \left( \frac{\partial^{\alpha-1} u_n}{\partial \xi^{\alpha-1}} - \left( x^2 + y^2 + z^2 \right) \right) \left( d\xi \right)^\alpha. \]

To get the iteration, we start with an initial approximation as

\[ u_0(x, y, z, t) = (x^2 + y^2 - z^2) t + (x^2 + y^2 + z^2) \frac{t^\alpha}{\Gamma(1 + \alpha)}, \]

\[ u_1(x, y, z, t) = (x^2 + y^2 - z^2) \frac{t^{1+\alpha}}{\Gamma(2 + \alpha)} + (x^2 + y^2 + z^2) \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)}, \]

\[ u_2(x, y, z, t) = (x^2 + y^2 - z^2) \frac{t^{1+2\alpha}}{\Gamma(2 + 2\alpha)} + (x^2 + y^2 + z^2) \frac{t^{3\alpha}}{\Gamma(1 + 3\alpha)}, \]

\[ u_3(x, y, z, t) = (x^2 + y^2 - z^2) \frac{t^{1+3\alpha}}{\Gamma(2 + 3\alpha)} + (x^2 + y^2 + z^2) \frac{t^{4\alpha}}{\Gamma(1 + 4\alpha)}, \]

and so on.

The series solution is given as

\[ u(x, y, z, t) = (x^2 + y^2) \left[ t^\alpha \frac{t^\alpha}{\Gamma(1 + \alpha)} + \frac{t^{1+\alpha}}{\Gamma(2 + \alpha)} + \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)} + \frac{t^{1+2\alpha}}{\Gamma(2 + 2\alpha)} + \ldots \right] \]

\[ + z^2 \left[ -t^\alpha \frac{t^\alpha}{\Gamma(1 + \alpha)} - \frac{t^{1+\alpha}}{\Gamma(2 + \alpha)} - \frac{t^{2\alpha}}{\Gamma(1 + 2\alpha)} - \frac{t^{1+2\alpha}}{\Gamma(2 + 2\alpha)} - \ldots \right]. \]

(29)

The exact solution \((\alpha = 2)\) in Equation (29) was found to be [22]

\[ u(x, y, z, t) = -(x^2 + y^2 + z^2) + \left( x^2 + y^2 \right) e^t + z^2 e^{-t}. \]
5. Conclusion

In this paper, we carefully proposed a reliable modification by considering the variational iteration method (VIM) to deal with the fractional differential equation within initial and boundary conditions. The objective of this paper is to apply the VIM via modified Riemann-Liouville fractional derivative. As shown in the examples of this paper, the proposed method is a powerful procedure for solving linear and nonlinear equations of fractional order. The simplicity and also easy to apply in programming are two special features of this method. It may be concluded that this method is more powerful and efficient than before in finding analytical as well as numerical solutions for a wide class of linear and nonlinear fractional differential equations.

References


